Math Olympiads Training Problems

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Abstract

This book is a translated into English extended and significantly added version of author's brochures "Guidelines for teachers of mathematics to prepare students for mathematical competitions" published at 1988 in Odessa.

Preface

This book is a translation into English of my brochures "guidelines for teachers of mathematics to prepare students for mathematical competitions" published 1988 year in Odessa.

More precisely it is corrected and significantly added version of this brochure. In comparison with the first original edition with solutions only to 20 problems from 112 problems represented there this new edition significantly replenished with new problems (around 180 problems).

And now all problems are accompanied by solutions which at different times done by the author of this book (sometimes multivariants and with the analysis and generalizations). Also, unlike the previous edition, all problems are grouped into the corresponding sections of mathematics.

Part I

Methodology Introduction

It makes no sense to repeat what has already been said about the usefulness and expediency of mathematical olympiads of different levels.Therefore, let us dwell on the issues that naturally arise in connection with the Olympiads, in particular, with olympiads of high level, -issues of preparation to Mathematical Competitions

The main question: Is it necessary such preparation?

It's not a secret that students who are able to solve the problems offered at these Olympiads, sufficiently gifted mathematically, have more advanced mathematical techniques and a number

of useful qualities, including the ability to self-organize and independent work.That is, and so good.?

But there is a fact of very serious preparation by level and by time, for participation in international mathematical Olympiads. Is known significant advantages of participants in the Olympiads, students of schools and classes, in which mathematics is taught in a larger volume and with greater depth.

Finally, the more capable a student is, the more important and difficult is to ensure the growing process of improving and systematizing mathematical education, which should include not only the knowledge of concrete facts, but what is more important, the ways of their formation (with the need to include their proofs), intensive practical work with solving non-standard and nonaddressed problems, that is everything that forming a culture of mathematical thinking.

(Culture of mathematical thinking:

-Discipline of thinking, algorithmic thinking, observation, ability to analysis, generalizations, the ability to build mathematical models, to choose a convenient language description of the problem situation .... (the list can be continued)).

The existing system of teaching mathematics in no way contributes to readiness of the student to solve unconventional, nonstandard problems of the Olympiad character. If all this happens, it is not thanks to this system, but contrary to it.

The main reason is that the goal of traditional school education is a certain an admissible minimum of knowledge, limited by the amount of hours, the program, its quantitative and qualitative composition and certainly the teaching methodology based mainly on the memorization of facts and means for execution of algorithmized instructions aimed at solving exclusively typical problems.

If within the framework of this system the student faithfully complies with all the requirements, and limited by this, then his success isn't sufficiently guaranteed. But this is not the main thing.

The main thing is that the creative attitude to mathematics will be hopelessly lost. And if this does not happen in some cases, it is only thanks to the personality of the student and the personality of the teacher that have fallen in the state of resonance.

It's no secret that the assurances of the organizers of the Olympiads that problems do not go beyond of school curriculum to put it mildly, distort the real state of things.

That is, formally they do not sin against the truth, at least so, how much, say, as the editor of the book, which write in the annotation, that for its reading not necessary to have no preliminary information, except for the developed mathematical thinking.

But the latter is already a result of a preparation of very long and intencive and varied, the result of systematic training aimed at developing thinking nonstandard, but logically disciplined.

The essence of this statement will become clear after the complete list of what is know the ordinary student (a student which is in full compliance with the program).

Even in class programs with in-depth study of mathematics, much of this in the following list is missing.

So, what does not know (or know insufficiently) a ordinary school student :

1. Algebraic and analytic technic.

2. Method of mathematical induction at the level of well-developed technique use it in different and, preferably, non-standard situations;

3. The theory of divisibility is, in a volume different from residual, vague representations of a high-school student about knowledge, which was casually received in middle school

4. The algebra of polynomials including the theory of divisibility of polynomials

5. Basic classic inequalities and their applications.

6. Integer and fractional parts. Properties and applications.

7. Technique of solving systems of inequalities in integer numbers and effective representation of integer multidimention domains;

8. Technique of summation, including summation by multidimention domains.

9. Sequences -different ways of their definition (including recursive definition and generating functions) and elementary methods of solving certain classes of recurrence relations and their applications in the theory of divisibility, summation, combinatorics and so on.

10. The Dirichlet principle.

11. Method of invariants.

12. Techniques of elementary (without derivatives) solving extremal problems, especially with many variables.

13. Solving equations in two or more unknowns in integers and especially in non-negative integers.

14. Sequence analysis (boundedness, monotonicity, limit theory, including theoretical and practical basis, and basic limits). .

To this list it is necessary to add the lack of the ability to solve non-standard, nonaddressed problems. An unconventional, unexpected problem should be classified, understood, reformulated,

simplified, immersed in a more general problem, or treated by special cases and identified the main theoretical tools needed to solve it. The usual work of a student is simple. Here is a chapter, here is a problem to this chapter. Search, recognition work is minimal and the emphasis is exclusively on the robustness of standard algorithms, that is, the minimum problematic level which is the only truly developing thinking factor.

And if thinking does not develop, then it degrades and even ideal diligence can not be a compensation for this loss, accompanying such approach to math education. Thus, a consequtive change of topics, not backed by "no-address" problems does not allow you to achieve the desired effect.

That is, it is necessary that in each topic there are problems that can be solved by using some a previously unknown combination of formally known theoretical propositions from the preceding material.

A participant in the Olympiad needs some psychological qualities that also require training and preparation, either special or spontaneous, accompanying the solution of non-standard problemss in conditions of limited time.

This ability to quickly and deeply focus on a specific problem, quickly relax and switch to another task from any previous emotional state depending on luck or failure.

Required sufficiently rich associative thinking and trained memory, allowing to carry out the associative search for necessary means to solve the problem.

And most importantly, to learn to "misunderstand", that is, to face a problem in which there is nothing to grab on, there is no (at first impression) ready-made approaches to its solution, calmly analyze it to look for something familiar and similar to what you know, consider special cases (reduction), generalize(induction), investigate the problem in limiting cases, introduce additional conditions that simplify the situation, accumulate experimental material.

For a mathematician, a difficult problems is to take height, to overcome not only intellectual barrier but also complexes, fears. Thus, are important the methodical settings of the type: "How to solve the problem?"; psychological attitudes: reaction to the shock of "misunderstanding," the creation of comfortable zones, the ability to relax, adjust to the problems, focus, quickly and deeply dive into it, that is requirements for the student's psychological status.

But the psychological and methodological qualities can only be developed by a large amount of work to solve non-standard problems, with the subsequent analysis of methodological, psychological and

especially technical and ideological aspects, with the formation of generalizing settings, which is also the goal of special training for students.

It often happens that children who are capable of creative work are not able to work in a sporting situation, which of course affects their "sports" results, but does not detract from their ability to mathematical creativity, which is by essence isn't a sports match.

However, as in other areas of human activity, people often bring sports excitement in mathematic, turning it into a competition of minds. By itself it is not a negative quality, but rather useful, developing

motive, under condition that the mathematics by itself is not reducible to one more kind of sports competition.

It should not be forgotten that the Math Olympiad is not an aim in itself, but a training ground on which many qualities necessary for the future researcher are being perfected, such as perseverance,

will, technique, knowledge, skills, reaction to practical situations, thinking.

The list of problems given in this book does not in any way pretend to be complete, but it is quite representative for such sections of school mathematics as arithmetic, algebra and analysis.

The absence of geometric problems proper is caused by the desire to restore the balance in the evidence base to school mathematics.

Traditionally, the concept of proof, the methods of proof, the level of rigor, the axiomatic approach - what we call abstract thinking is basically formed in the course of geometry, that is, a region

closer to sensory perception than algebra.

Arithmetic turns out to abandoned wasteland, somewhere in the backyard of mathematical construction, and algebra is reduced to a set of formulas and rules that need to be remembered and applied.

At the same time, an insignificant part of students informally accepts and understands the level of rigor and evidence in geometry and is able to transfer the acquired thinking technique independently

to other mathematical areas.

For the others, the geometry - not motivated and it is unclear for what sins the punishment by jesuitically sophisticated logic and for some reason mostly "from the opposite", resulting in a

false and unimaginable premise to the almost illusive stunt of drawing a black rabbit from a black hat, "what was required to prove ".

The loss in the geometry of its naturalness, the departure from the exposition of it in the school at the level of Euclid, Kiselev, Kokseter did not bring desirable effect in the plane of it's modernization, rigor or deep understanding, since geometry is not the object on which the axiomatic method, usually accompanied with extreme formalism don't bring true efficiency.

This is particularly true for the introductory courses specific to the school. And although historically, it was in geometry, the axiomatic method showed us its methodological power, geometry

is not at all the only reason for its primary demonstration.

Speaking of this, I in no way deny the use of the elements of the axiomatic method and rigorous proofs in the exposition of geometry, moreover, I consider their use necessary and beneficial.

I am only expressing the doubt, confirmed by my teaching practice, in the appropriateness and effectiveness of primacy in the use of the axiomatic method and proofs in geometry.

It is not with geometry that one has to start the introduction of mathematical formalism, which comes into conflict with the visibility which inherent to geometry.

Due to the structural wealth of geometry, its axiomatics are voluminous and combinatorially saturated and, although grown by abstraction from sensory images, nevertheless do not live well with them.

Rather, they are poorly compatible with the level of rigor and formalism of thinking, which is the inevitable companion of the axiomatic method.

And the roots of this in the psychology of perception and thinking.

It is known that it is most difficult to prove or disprove the apparently obvious, visual:

"The sun revolves around the Earth," "The Earth is flat," "A straight line that crosses one side of the triangle and does not pass through the vertices of a triangle will cross exactly one of its sides."

Hence the Greeks instead of proof drawing and saying "look!".

History begins with geometry, the school copies history, although it is known that when the path is passed, it is not the shortest and most effective.

At the same time, the Peano axioms for natural numbers, the theorems in the divisibility theory, the axioms of various algebraic structures that are essentially a subset of geometric axiomatics are much simpler (combinatorial complexity), less reducible to sensory images, and therefore their use is methodologically more justified.

[Advanced algebraic base assuming free possession of the symbolic transformative technique, systematic proof of all the theoretical facts that make up the qualitative and the computational basis of what is commonly called school algebra (see items 1,2,3, ... of the list of the above) - this should, in my opinion, precede the rest of the school mathematics, including geometry.]

But in school, these topics are taboo. Suffers from such a one-sidedness all math subjects algebra, and arithmetic, and geometry, that is, all of school mathematics and not only.

Mathematics is one, its means are universal - this is the ideological basis on which mathematics education should be carried out. And methodological one-sidedness is unacceptable.

And, finally, the implantation of mathematical methodology into consciousness should should be implemented by the way which is most motivated psychologically.

The lack of habit of abstract reasoning at the level of the proofs of theorems in arithmetic and algebra, in contrast to the intensive theoretical foundation in geometry, subsequently creates a

considerable obstacle in the ability to find arithmetic (algebraic) means, to dispose of them with the same rigor and thoroughness as is customary in geometry.

Quite often the idea of the non-standard and complexity of the arithmetic problem is related precisely to the absence of a completely elementary and essential sequential theoretical basis related to arithmetic of natural, integer and numbers in general, which forms, in addition to everything, is the foundation of mathematical analysis. That is, the non-standard nature of the problem in such cases is equivalent to non-informedness.

In these cases, the situation becomes ambiguous, because, on the one hand, the lack of specific knowledge-tools requires its spontaneous invention in the conditions of the Olympiad, and it is more complicated than choosing the necessary combination of already known tools and technology for its purposeful use, and on the other hand for an informed student solution of the such problem basically becomes a matter of technique.

Thus, the olympiad (sports) value of such problems is doubtful. This does not, however, diminish their possible educational value. But let us leave that on the conscience of the composers of the Olympiad problems and consider the positive aspect of this situation, which consists in motivation of the student in additional technical and theoretical equipment, which ultimately brings him to a higher level, and allows expanding the problem area, then there is more complex problems, the solution of which already depends entirely from recognizing ability of the participant of the Olympiad and his ingenuity in the use of already known means. In this way, both the stimulation of mathematical education and the escalation of thinking take place.

In the author's opinion, the problems presented in the following sections will convincingly argued that was saying in the introduction.

Remark.

1.Abbreviation **n-Met. Rec.**(**Methodical recommendations**) means that the problem originally has number n in the author's brochures "Guidelines for teachers of mathematics to prepare students for mathematical competitions" published at 1988 in Odessa.

2.Abbreviation **MR** means **Mathematical Reflections** – **AwesomeMath**;

3. Abbreviation **ZK** means **Zadachnik Kvanta;**

4. Abbreviation **SSMJ** means **School Science and Mathematics Association Journal**

5. Also, if problem marked by sign ★ it means that the problem was proposed by author of this book.

Part II

Problems

1 Divisibility.

Problem 1.1 (6-Met.Rec.)

Find all n such that $1\underbrace{44\ldots4}_{n\ times}$ is the perfect square.

Problem 1.2 (8-Met.Rec.)

Prove that number $385^{1980}+18^{1980}$ isn't a perfect square.

Problem 1.3 (9-Met.Rec.)

Let $f(x)=x^3-x+1$. Prove that for any natural a numbers $a,\ f(a),\ f(f(a)),$ ...,are pairwise coprime.

Problem 1.4(23-Met.Rec.)

Find the largest natural x such that $4^{27}+4^{1000}+4^x$ is a perfect square.

Problem 1.5(24-Met.Rec.)

Prove that 5^n-4^n for any natural $n>2$ isn't perfect square.

Or,

Prove that set $\{5^n-4^n \mid n>2\}$ is free from squares.

Problem 1.6(25-Met.Rec.)

a) Prove that set $\{2^n+4^n \mid n\in\mathbb{N}\}$ is free from squares;

b) Find all non negative integer n and m for which 2^n+4^m is perfect square.

Problem 1.7(26-Met.Rec.)

Find all $n\in\mathbb{N}$ such that 3^n+55 is a perfect square.

Problem 1.8 (27-Met. Rec.)
Prove that the following number is composite for any natural n:
a) $a_n := 3^{2^{4n+1}} + 2$;
b) $b_n := 2^{3^{4n+1}} + 3$;
c) $c_n := 2^{3^{4n+1}} + 5$.

Problem 1.9(28-Met. Rec.)
Prove that $5^n - 1$ isn't divisible by $4^n - 1$ for any $n \in \mathbb{N}$.

Problem 1.10(297-Met. Rec.)
Let a, b, c, d be natural numbers such that $ab = cd$. Prove that for any natural n number $a^{2n} + b^{2n} + c^{2n} + d^{2n}$ is composite.

Problem 1-11(30-Met. Rec.)
Prove that $5^{3^{4m}} - 2^{2^{4n+2}}$ is divisible by 11 for any natural m, n.

Problem 1.12(32-Met. Rec.)
Is there a number whose square is equal to the sum of the squares of 1000 consecutive integers?

Problem 1.13(33-Met. Rec.)
Let n be natural number such that $2n + 1$ and $3n + 1$ are perfect squares. Prove that n is divisible by 40.

Problem 1.14(34-Met. Rec.)
Is it possible that sum of digits of a natural number which is a perfect square be equal 1985?

Problem 1.15
Find all non negative integer n and m for which $2^n + 4^m$ is perfect square.

Problem1.16(37-Met. Rec.)
Prove that:
a) $n!$ isn't divisible by 2^n.
b) $ord_p\left(((p-1)\,n)!\right) \le n + ord_p\,(n!)$.
c) $(n!)! \ge (((n-1)!)!)^n$
d) $ord_p\left(\dfrac{(pn)!}{n!}\right) = n$.
e) $(n!)! \ge ((n-1)!)^{n!}$.

Problem 1.17(38-Met. Rec.)
Prove that:
a) $(n!)! \,\vdots\, (n!)^{(n-1)!}$;

b) $(n!)! \vdots ((n-1)!)!^n$;

c) $(n^n)! \vdots n!^{n^{n-1}}$;

d) $\left(n^2\right)! \vdots (n!)^n$;

e) $\left(n^{m+k}\right)! \vdots (n^m)!^{n^k}$;

f) $(n \cdot m)! \vdots (n!)^m$;

g) $\dfrac{(2n)!}{n!\,(n+1)!}$ is integer.

h) $(n+1)(n+2)\dots(n+k) \vdots k!$ for any $n, k \in \mathbb{N}$.

Problem 1.18(41-Met. Rec.)
Find all natural number n such that remainder from division
$S_n = 1 + 2 + \dots + n$ by 5 equal 1.

Problem 1.19(123-Met. Rec.)
Show that the next integer above $\left(\sqrt{3}+1\right)^{2n}$ is divisible by 2^{n+1},i.e.
$\left\lceil\left(\sqrt{3}+1\right)^{2n}\right\rceil \vdots 2^{n+1}$.Show that there are infinitely many $n \in \mathbb{N}$ for which
$\left\lceil\left(\sqrt{3}+1\right)^{2n}\right\rceil$ not divisible by 2^{n+2}.

2 Diophantine equation.

Problem 2.1(22-Met. Rec.)

Find all integer x such that $\dfrac{3x - \sqrt{9x^2 + 160x + 800}}{16}$ is integer.

Problem 2.2(35-Met. Rec.)
Prove that equation $x^2 - 2xy = 1978$ have no sulutions in integers.

Problem 2.3(47-Met. Rec.)
Prove that if numbers $n, m \in \mathbb{N}$ satisfy to equality $2m^2 + m = 3n^2 + n$
then numbers $m - n, 2m + 2n + 1, 3m + 3n + 1$ are perfect squares.

Problem 2.4(42-Met. Rec.)
Find all integer solutions of equation $x^3 - 2y^3 - 4z^3 = 0$ (excluding
trivial solution $x = y = z = 0$).

Problem 2.5(43-Met. Rec.)
How many natural solutions have equation $2x^3 + y^5 = z^7$?

Problem 2.6(44-Met. Rec.)
Prove that equation $x^3 + y^3 + z^3 + t^3 = u^4 - v^4$ has infinitely many
solutions in natural x, y, z, t, u, v.

Problem 2.7(45-Met. Rec.)
How many natural solutions has equation $x^4 + y^6 + z^{12} = t^4$?

Problem 2.8(46-Met. Rec.)
Prove that for any given integer t the following equations have no integer solutions:
a) $x^3 + y^3 = 9t \pm 4$;
b) $x^3 + y^3 = 9t \pm 3$;
c) $x^3 + y^3 + z^3 = 9t \pm 4$;
d) $x^3 + 117y^3 = 5$.

Problem 2.9(50-Met. Rec.)
Let a be integer number such that $3a = x^2 + 2y^2$ for some integer numbers x, y.
Prove that number a can be represented in the same form , that is there is integers u, v that $a = u^2 + 2v^2$.

Problem 2.10(40-Met. Rec.)
Find conditions for irreducible fractions $\dfrac{a}{b}$ and $\dfrac{c}{d}$ that provide silvability of equation $y = x^2 + \dfrac{a}{b}x + \dfrac{c}{d}$ in integer x, y.(that parabola contain at least one (then infinitely many) points $M(x, y)$ with integer x, y.

★ Problem 2.11(3932, CRUX)
Prove that for any natural numbers x, y satisfying equation
$x^2 - 14xy + y^2 - 4x = 0$
holds $\gcd^2(x, y) = 4x$.

Problem 2.12(54-Met. Rec.)
The store has a sealant in boxes of 16lb, 17lb, 21lb. How some organization can get without opening boxes 185 lb of sealant and so, that the number of boxes will be smallest?

Problem 2.13(55-Met. Rec.)
Find the number of non-negative integer solutions of equation
$5x + 2y + z = 10n$ in term of given natural n.

3 Integer and fractional parts.

Problem 3.1 (56-Met. Rec.)
Find $\left[\left(\sqrt[3]{2} + \sqrt[3]{4}\right)^3\right]$.

Problem 3.2 (57-Met. Rec.)
Simplify

a) $\left[\left(\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}\right)^2\right]$;
b) $\left[\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}\right]$.

Problem 3.3 (59-Met. Rec.)
Solve equation $\{x\}+\left\{\dfrac{1}{x}\right\}=1, x\in\mathbb{R}$.

Problem 3.4 (60-Met. Rec.)
Prove equality $\sum\limits_{a=2}^{n}[\log_a n]=\sum\limits_{b=2}^{n}[\sqrt[b]{n}]$.

★Problem 3.5 (3095, CRUX)
Let a, b, c, p, and q be natural numbers. Using $\lfloor x\rfloor$ to denote the integer part of x, prove that

$$\min\left\{a,\left\lfloor\frac{c+pb}{q}\right\rfloor\right\}\leq\left\lfloor\frac{c+p(a+b)}{p+q}\right\rfloor.$$

Problem 3.6 (10-Met. Rec.)
Prove that:
a) For any $n\in\mathbb{N}$ holds inequality $\left\{n\sqrt{2}\right\}>\dfrac{1}{2n\sqrt{2}}$;
b) For any $\varepsilon>0$ there is $n\in\mathbb{N}$ such that $\left\{n\sqrt{2}\right\}<\dfrac{1+\varepsilon}{2n\sqrt{2}}$.

Problem 3.7 (11-Met. Rec.)
Let $n\in\mathbb{N}$ isn't forth degree of natural number. Then

$$\{\sqrt[4]{n}\}>\frac{1}{4n^{3/4}}.$$

★Problem 3.8 (J289,MR)
For any real $a\in[0,1)$ prove the following identity

$$\left[a\left(1+\left[\frac{1}{1-a}\right]\right)\right]+1=\left[\frac{1}{1-a}\right].$$

Problem 3.9 (118-Met. Rec.)
For arbitrary natural $m\geq 2$ prove that $\left\lfloor\left(m+\sqrt{m^2-1}\right)^n\right\rfloor$ is odd number for any natural n.

★ Poblem 3.10 (W16, J.Wildt IMO 2017)
For given natural $n>1$ find number of elements in image of function

$$k\mapsto\left[\frac{k^2}{n}\right]:\{1,2,...,n\}\longrightarrow\mathbb{N}\cup\{0\}.$$

4 Equations, systems of equations.

★Problem 4.1(90-Met. Rec.)(Generalization of M703* Kvant)
Solve the system of equations.

$$\begin{cases} (q+r)(x+1/x) = (r+p)(y+1/y) = (p+q)(z+1/z) \\ xy+yz+zx=1 \end{cases}$$

where p,q,r are positive real numbers.

Problem 4.2 (91-Met. Rec.)

Solve the system of equations

$$\begin{cases} 2x+x^2y=y \\ 2y^2+y^2z=z \\ 2z^2+z^2x=x \end{cases}.$$

Problem 4.3 (92-Met. Rec.)

Solve the system of equations:

$$\begin{cases} x-y=\sin x \\ y-z=\sin y \\ z-x=\sin z \end{cases}.$$

Problem 4.4 (93-Met. Rec.)

Solve the system of equations:

$$\begin{cases} x_1+x_2+...+x_n=1 \\ x_1^2+x_2^2+...+x_n^2=\dfrac{1}{n} \end{cases}.$$

Problem 4.5 (94-Met. Rec.)

Solve the system of equations:

a) $\begin{cases} x^2+y^2+z^2=1 \\ x+y+az=1+a \end{cases}, a \geq \dfrac{1}{2};$

b) $\begin{cases} x+y+z=a \\ \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{a} \end{cases}, a \neq 0.$

Problem 4.6 (95-Met. Rec.)

Given that

$$\begin{cases} x+y+z=2 \\ xy+yz+zx=1 \end{cases}.$$

Prove that $x,y,z \in [0,4/3]$.

Problem 4.7(96-Met. Rec.)

Solve the system of equations:

$$\begin{cases} 2(\cos x-\cos y)=\cos 2x\cos y \\ 2(\cos y-\cos z)=\cos 2y\cos z \\ 2(\cos z-\cos x)=\cos 2z\cos x \end{cases}.$$

5 Functional equations and inequalities

Problem 5.1 (97-Met. Rec.)

Find all functions defined on $\mathbb{R}$ such that:

a) $f(x^2)-(f(x))^2 \geq 1/4$ and $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$;

b) $f(x) \leq x$ for any $x \in \mathbb{R}$ and $f(x+y) \leq f(x)+f(y)$ for any $x,y \in \mathbb{R}$.

Problem 5.2 (99-(Met. Rec.)

Function $f(x)$ defined on $[0,1]$ and satisfies to equation
$f(x+f(x)) = f(x)$
for any $x \in [0,1]$ .Prove that $f(x) = 0$ for all $x \in [0,1]$.

Problem 5.3 (100-Met. Rec.)
Find all continuous on $\mathbb{R}$ functions f suth that
$f(x)f(y) - xy = f(x) + f(y) - 1$
holds for any $x, y \in \mathbb{R}$.

Problem 5.4 (101-Met. Rec.)
Let $n \in \mathbb{N} \diagdown \{1\}$. Find all defined on $\mathbb{R}$ functions f such that
$nf(nx) = f(x) + nx$ for any $x \in \mathbb{R}$ and f is continuous in $x = 0$.

Problem 5.5 (14-Met. Rec.)
Prove that there is no function $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous on $\mathbb{R}$ such that
$$f(x+1)(f(x)+1)+1 = 0.$$

Problem 5.6 (15-Met. Rec.)
For any given $n \in \mathbb{N}$ find all functions $f : \mathbb{N} \longrightarrow \mathbb{R}$ such that
$f(m+k) = f(mk-n)$, $m, k \in \mathbb{N}$ and $mk > n$.

★Problem 5.7 (U182,MR)
Find all continuous on $[0,1]$ functions f such that $f(x) = c$, if $x \in \left[0, \dfrac{1}{2}\right]$ and $f(x) = f(2x-1)$ if $x \in \left(\dfrac{1}{2}, 1\right]$,where c is given constant.

6 Recurrences.

Problem 6.1 (4-Met. Rec.)
Let p is some natural number. Prove, that exist infinitely many pairs (x, y) of natural numbers such, that $\dfrac{x^2+p}{y}$ and $\dfrac{y^2+p}{x}$ are integer numbers.

Problem 6.2 (5-Met. Rec.)
Let sequence is defined recursively as follow:
$$a_{n+3} = \frac{a_{n+1}a_{n+2}+5}{a_n}, n \in \mathbb{N} \text{ and } a_1 = a_2 = 1, a_3 = 2.$$
Prove that all terms of this sequence are integer numbers.

Problem 6.3 (16-Met. Rec.Problem 5, Czechoslovakia, MO 1986)
Sequence of integer numbers $a_1, a_2, ..., a_n, ...$ defined as follows:
$a_1 = 1, a_{n+2} = 2a_{n+1} - a_n + 2, n \in \mathbb{N}$.
Prove that for any $n \in \mathbb{N}$ there is such $m \in \mathbb{N}$ that $a_n a_{n+1} = a_m$.

Problem 6.4 (17-Met. Rec.)
Prove that if sequence $(a_n)_{n\geq 1}$ satisfy to recurrence
$a_{n+2} = a_{n+1}^2 - a_n, n \in \mathbb{N}$
with initial conditions $a_1 = 39, a_2 = 45$ then infinitely many terms of this sequence is divisible by 1986.

Problem 6.5* (31-Met. Rec.)
Given a quad of integer numbers (a, b, c, d) such that at least two of them are different.
Starting from this quad we create new quad
$(a_1, b_1, c_1, d_1) = (a - b, b - c, c - d, d - a)$.
By the same way from quad (a_1, b_1, c_1, d_1) we obtain quad (a_2, b_2, c_2, d_2) and so on...
Prove that at least one from the numbers $a_{100}, b_{100}, c_{100}, d_{100}$ bigger than 10^9.

Problem 6.6* (19-Met. Rec)
Let $a > 1$ is natural number. Sequence $a_1, a_2, ..., a_n,$...is defined recursively

$$\begin{cases} a_1 = a \\ a_n = a^n - \sum\limits_{t|n,\ t<n} a_t \end{cases}.$$

Prove that $a_n \vdots n$ for any $n \in \mathbb{N}$ (a_n divisible by n for any $n \in \mathbb{N}$).

Problem 6.7* (G.Demirov, Matematika 1989,No.7 ,p.34, Bolgaria)
Let sequence (a_n) defined by the recurrence
$a_{n+2} = a_{n+1}a_n - 2(a_{n+1} + a_n) - a_{n-1} + 8, n \in \mathbb{N}$ with initial conditions $a_0 = 4, a_1 = a_2 = (a^2 - 2)^2$, where $a \geq 2$.
Prove that for any $n \in \mathbb{N}$ expression $2 + \sqrt{a_n}$ is a square of some polinomial of a.

Problem 6.8
Find general term of the sequence:
a) $a_{n+1} = \dfrac{1}{27}\left(8 + 3a_n + 8\sqrt{1 + 3a_n}\right), a_1 = \dfrac{8}{3}$;
b) $a_{n+1} = \dfrac{1}{16}\left(1 + 4a_n + \sqrt{1 + 24a_n}\right), a_1 = 1$.

Problem 6.9*
Let sequence (a_n) be defined by equation $\left(\sqrt{2} - 1\right)^n = \sqrt{a_n + 1} - \sqrt{a_n}$.
a) Find recursive definition for (a_n) and prove that a_n is integer for all natural n;
b) Let $t_n := \sqrt{2a_n(a_n + 1)}$. Find recursive definition for (t_n) and prove that t_n is integer for all natural n.

Problem 6.10.(Proposed by S. Harlampiev, Matematika 1989, No.2, p,43, Bolgaria)

Let sequence defined by recurrence
$$a_{n+2} = \frac{2a_{n+1} - 3a_{n+1}a_n + 17a_n - 16}{3a_{n+1} - 4a_{n+1}a_n + 18a_n - 17}, n \in \mathbb{N} \cup \{0\}$$
with initial conditions $a_0 = a_1 = 2$.
Prove that a_n for any $n \in \mathbb{N} \cup \{0\}$ can be represented in the form $1 + \dfrac{1}{m^2}$ where $m \in \mathbb{N}$.

Problem 6.11*. (Proposed by Bulgaria for 1988 IMO)

Let $a_0 = 0, a_1 = 1, a_{n+1} = 2a_n + a_{n-1}, n \in \mathbb{N}$. Prove that $a_n \vdots 2^k \iff n \vdots 2^k$.

7 Behavior(analysis) of sequences

Problem 7.1 (104-Met.Rec)
For natural $n \geq 3$ let $a_1, a_2, ..., a_n$ be real numbers such that $a_1 = a_n = 0$ and $a_{k-1} + a_{k+1} \leq 2a_k, k = 2, ..., n-1$. Prove that $a_k \geq 0, i = 1, 2, ..., n$.

Problem.7.2 (105-Met. Rec.)
a) Let $a_1 = 1$ and $a_{n+1} = a_n + \dfrac{1}{a_n}, n \in \mathbb{N}$. Prove that $14 < a_{100} < 18$.
Find lower and upper bounds for a_n.**(Problem 7 from all Soviet Union Math Olympiad,1968)**
b) Let $a_1 = 1$ and $a_{n+1} = a_n + \dfrac{1}{a_n^2}, n \in \mathbb{N}$.
i.Prove that (a_n) unbounded.
ii. $a_{9000} > 30$;
iii.★ find good (assimptotic) bounds for (a_n).

Problem 7.3 (106-Met. Rec.)
Find all values of a,such that sequence $a_0, a_1, ..., a_n, ...$ defined as follows $a_0 = a, a_{n+1} = 2^n - 3a_n, n \in \mathbb{N} \cup \{0\}$ is increasing sequence.

Problem 7.4 (107-Met. Rec.)
Known that sequence $a_1, a_2, ..., a_n, ...$ satisfy to inequality
$a_{n+1} \leq \left(1 + \dfrac{b}{n}\right) a_n - 1, n \in \mathbb{N}$, where $b \in [0, 1)$.
Prove that there is n_0 such that $a_{n_0} < 0$.

Problem 7.5* (**109-Met.Rec.) (Team Selection Test, Singapur).**
Let $n \in \mathbb{N}$, $a_0 = \dfrac{1}{2}$ and $a_{k+1} = a_k + \dfrac{a_k^2}{n}, k \in \mathbb{N}$. Prove that
$$1 - \frac{1}{n} < a_n < 1.$$

Problem 7.6 (110-Met. Rec.)
Find $\lim\limits_{n\to\infty}\left\{\left(2+\sqrt{3}\right)^{n}\right\}$.

Problem 7.7 (111 -Met. Rec.)
a) Let sequence (x_n) satisfy to recurrence $x_{n+1}=x_n(1-x_n)$, $n\in\mathbb{N}\cup\{0\}$ and $x_0\in(0,1)$. Prove that $\lim\limits_{n\to\infty} nx_n=1$;
b) Let sequence (x_n) satisfy to recurrence
$x_{n+1}=x_n^2-x_n+1$ and $x_1=a>1$.
i. Find $\sum\limits_{n=1}^{\infty}\dfrac{1}{x_n}$;
ii. Find $\left\lfloor\dfrac{x_{n+1}}{x_1x_2...x_n}\right\rfloor$.
c) Let sequence (x_n) satisfy to recurrence $x_n=0.5x_{n-1}^2-1, n\in\mathbb{N}$
with initial condition $x_0=\dfrac{1}{3}$.
Find $\lim\limits_{n\to\infty} x_n$.

Problem 7.8 (112-Met. Rec.)
Find $\lim\limits_{n\to\infty} x_n$ where $x_0=1/3, x_{n+1}=0.5x_n^2-1, n\in\mathbb{N}\cup\{0\}$.

Problem 7.9* Let $a_1=\dfrac{1}{2}$ and $a_{n+1}=a_n-na_n^2, n\in\mathbb{N}$.
a) Prove that $a_1+a_2+...+a_n<\dfrac{3}{2}$ for all $n\in\mathbb{N}$.
b*) Find "good" bounds for a_n,i.e. such two well calculating function $l(n)$ and $u(n)$such that $l(n)\le a_n\le u(n)$ for all n greater then some n_0 and $\lim\limits_{n\to\infty}\dfrac{a_n}{u(n)}=\lim\limits_{n\to\infty}\dfrac{a_n}{l(n)}=1$
(This equivalent to $\lim\limits_{n\to\infty}\dfrac{a_n}{l(n)}=\lim\limits_{n\to\infty}\dfrac{u(n)}{l(n)}=1$ or to
$\lim\limits_{n\to\infty}\dfrac{a_n}{u(n)}=\lim\limits_{n\to\infty}\dfrac{l(n)}{u(n)}=1$);
We call two function $l(n)$ and $u(n)$ asymtotically equal and write it $l(n)\sim u(n)$ if $\lim\limits_{n\to\infty}\dfrac{l(n)}{u(n)}=1$. Thus, function $l(n)$ and $u(n)$ is good bound
iff $l(n)\sim u(n)$ and $l(n)\le a_n\le u(n)$.
c) Determine asymptotic behavior of a_n ,i.e. find function asymptotically equal to a_n.
(or more simple question: Find $\lim\limits_{n\to\infty} n^2a_n$).

Problem 7.10
Let sequence (a_n) satisfy to recurrence $a_{n+1}=\dfrac{a_n^2-2}{2}, n\in\mathbb{N}$.Prove that:
i. If $a_1=1$ then (a_n) is bounded;
ii. If $a_1=3$ then (a_n) is unbounded.

Problem 7.11
Let sequence (a_n) defined by $a_1 = 1, a_{n+1} = \dfrac{3}{4}a_n + \dfrac{1}{a_n}, n \in \mathbb{N}$. Prove that:
i. (a_n) is bounded;
ii. Prove that $|a_n - 2| < \left(\dfrac{2}{3}\right)^n, n \in \mathbb{N}$.
Generalization: $a_{n+1} = pa_n + \dfrac{1}{a_n}$, $n \in \mathbb{N}$ for any given $p \in (0, 1)$.

Problem 7.12 (Bar-Ilan University math. olympiad, Israel).
Let $a_1 = 1, a_{n+1} = 1 + \dfrac{1}{a_n}, n \in \mathbb{N}$. Prove that there is real number b which for all $n \in \mathbb{N}$ satisfy inequality $a_{2n-1} < b < a_{2n}$.

Problem 7.13
Let $a_0 = 1$ and $a_{n+1} = \dfrac{a_n}{2} + \dfrac{1}{a_n}$ for $n = 0, 1, 2, ...$Prove that $\dfrac{2}{\sqrt{a_n^2 - 2}}$ is an integer for every natural n.

Problem 7.14(All Israel Math. Olympiad in Hayfa)
Given m distinct, non-zero real numbers $a_1, a_2, ..., a_m, m > 1$.
Let $A_r = a_1^r + a_2^r + .. + a_m^r$ for any natural $r \geq 1$.
Prove that for odd m inequality $A_r \neq 0$ holds for all r up to finite set of values r.

Problem 7.15*(#7,9-th grade,18-th All Soviet Union Math Olympiad,1984, Proposed by Agahanov N.H.)
Let $x_1 = 1, x_2 = -1$ and $x_{n+2} = x_{n+1}^2 - \dfrac{x_n}{2}, n \in \mathbb{N}$.
Find $\lim\limits_{n \to \infty} x_n$.

Problem 7.16
Given sequence of positive numbers (a_n) such that $a_{n+1} \leq a_n (1 - a_n)$.
Prove that sequence (na_n) is bounded.

Problem 7.17 (BAMO-2000)
Given sequence (a_n) such that $a_1 > 0$ and $a_n^2 \leq a_n - a_{n+1}, n \in \mathbb{N}$.
Prove that $a_n < \dfrac{1}{n}$ for all natural $n \geq 2$.

★**Problem 7.18 (SSMJ 5281)**
For sequence $\{a_n\}_{n \geq 1}$ defined recursively by

$$a_{n+1} = \frac{a_n}{1 + a_n^p} \text{ for } n \in \mathbb{N},\ a_1 = a > 0,$$

determine all positive real p for which series $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

Problem 7.19
Given $a_1 = 5, a_{n+1} = a_n^2 - 2, n \in \mathbb{N}$.

a) Find $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_1 a_2 ... a_n}$;
b) Find $\lim\limits_{n\to\infty} \left(\dfrac{1}{a_1} + \dfrac{1}{a_1 a_2} + ... + \dfrac{1}{a_1 a_2 ... a_n}\right)$.

Problem 7.20*
Let $a_1 = a$,where $a > 0, a_{n+1} = \dfrac{a_n}{1+\sqrt{a_n}}$, $n \in \mathbb{N}$.
a) Prove that sum $S_n = a_1 + a_1 + ... + a_n$ is bounded ;
b***)** ★ Find "good" bounds for a_n if $a_1 = 9$.
(Or, find asymptotic representation for a_n)
c) Find the $\lim\limits_{n\to\infty} n^2 a_n$.

★ Problem 7.21 (One asymptotic behavior) (S183)
Let sequence (p_n) satisfied to recurrence
$p_n = p_{n-1} - \dfrac{p_{n-1}^2}{2}, n = 1, 2, ...$ and $p_0 \in (0, 1)$.
Prove that $\dfrac{2}{n+\sqrt{n}+p+1} < p_n < \dfrac{2}{n+p}, n \in \mathbb{N},$ where $p := \dfrac{2}{p_0}$.

8 Inequalities and max,min problems.

Comparison of numerical expressions.

Problem 8.1 (81-Met. Rec.)
Determine which number is greater (here $\vee$ is sign of ineqeality $<$ or $>$ in unsettled state).
a) $31^{11} \vee 17^{14}$;
b) $127^{23} \vee 513^{18}$;
c) $53^{36} \vee 36^{53}$;
d) $\tan 34^\circ \vee \dfrac{2}{3}$;
e) $\sin 1 \vee \log_3 \sqrt{2}$;
f) $\log_{(n-1)} n \vee \log_n (n+1)$;
h) $100^{300} \vee 300!$;
g) $(n!)^2 \vee n^n$;
i) $\sqrt{2+\sqrt{3+\sqrt{2+...}}} \vee \sqrt{3+\sqrt{2+\sqrt{3+...}}}$ (n roots in each expression).
For any natural n compare two numbers $a_n = \sqrt{2+\sqrt{3+\sqrt{2+...}}}$ and $b_n = \sqrt{3+\sqrt{2+\sqrt{3+...}}}$ (each use n square root simbols).What is greater?

Proving inequalities

Problem 8.2 (Inequality with absolute value)
Let a, b, c be real numbers such that $a + b + c = 0$.Prove that

$|a \cdot b \cdot c| \leq \frac{1}{4} \max \left\{ |a|^3, |b|^3, |c|^3 \right\}.$

Problem 8.3 (69-Met. Rec.)
Let $x, y, z \geq 0$ and $x + y + z \leq \frac{1}{2}$. Prove that
$(1 - x)(1 - y)(1 - z) \geq \frac{1}{2}.$

Problem 8.4 (Problem 6 from 6-th CGMO, 2-nd day,2007).
For nonnegative real numbers a, b, c with $a + b + c = 1,$ prove that

$$\sqrt{a + \frac{(b-c)^2}{4}} + \sqrt{b} + \sqrt{c} \leq \sqrt{3}.$$

Problem 8.5 (70-Met. Rec.)
Prove that for any positive real $a_1, a_2, ..., a_n, n \geq 3$ holds inequality
$\sum_{cyc}^{n} \frac{a_1 - a_3}{a_2 + a_3} \geq 0$ (Or, $\sum_{i=1}^{n} \frac{a_i - a_{i+2}}{a_{i+1} + a_{i+2}} \geq 0$ $(a_{n+1} = a_1, a_{n+2} = a_2)$.

Problem 8.6 (71-Met. Rec.)
For any positive real a, b, c prove inequality
$\frac{a^3}{a^2 + ab + b^2} + \frac{b^3}{b^2 + bc + c^2} + \frac{c^3}{c^2 + ca + a^2} \geq \frac{a + b + c}{3}.$

Problem 8.7 (72-Met. Rec.)
Prove that any nonnegative real a, b, c holds inequlity
$a^5 + b^5 + c^5 \geq abc(ab + bc + ca).$

Problem 8.8 (74-Met. Rec.)
Prove that $\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} \leq \sqrt{21}$ if $a, b, c > -\frac{1}{4}$ and $a + b + c = 1.$

Problem 8.9 (75-Met. Rec.)
Prove that $(x_1 + x_2 + ... + x_n + 1)^2 \geq 4\left(x_1^2 + x_2^2 + ... + x_n^2\right), x_i \in [0, 1],$ $i = 1, 2, ..., n.$

Problem 8.10 (76-Met. Rec.)
Let x, y, z be positive real numbers. Prove that $x^3z + y^3x + z^3y \geq xyz(x + y + z).$

Problem 8.11 (77-Met. Rec.)(Oral test in MSU)
Solve inequality $\left(\frac{x_1y_1 + x_2y_1}{x_1y_1 + x_1y_2}\right)^{x_1} \left(\frac{x_2y_2 + x_1y_2}{x_2y_2 + x_2y_1}\right)^{x_2} \geq 1$ for real positive x_1, x_2, y_1, y_2.

Problem 8.12 (78-Met. Rec.)
Prove inequality
$\sqrt{2} + \sqrt{4 - 2\sqrt{2}} + \sqrt{6 - 2\sqrt{6}} + ... + \sqrt{2n - 2\sqrt{n(n-1)}} \geq \sqrt{n(n+1)}.$

Problem 8.13 (79-Met. Rec.)
Given that $a_1, a_2, ..., a_n$ are positive numbers and $a_1 + a_2 + ... + a_n = 1$.
Prove that

$$\sum_{k=1}^{n} a_k \sqrt{1 - \left(\sum_{i=1}^{k} a_i\right)^2} < \frac{4}{5}.$$

Problem 8.14 (84-Met. Rec.)
Let $a_1, a_2, ..., a_n$ be positive real numbers. Prove that

$$a_1a_2 + a_2a_3 + ... + a_{n-1}a_n + a_na_1 \le \begin{cases} \dfrac{(a_1 + a_2 + ... + a_n)^2}{n}, \text{ if } n = 2, 3 \\ \dfrac{(a_1 + a_2 + ... + a_n)^2}{4}, \text{if } n \ge 4 \end{cases}.$$

Problem 8.15 (85-Met. Rec.).Original setting.
Prove that for any numbers $a_1, a_2, ..., a_n \in [0, 2]$, $n \ge 2$ holds inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |a_i - a_j| \le n^2.$$

*More difficult variant of the problem:
Find max $\sum_{1 \le i < j \le n}^{n} |a_i - a_j|$, if $a_1, a_2, ..., a_n$ be any real numbers such that $|a_i - a_j| \le 2, i, j \in \{1, 2, ..., n\}$.

★ Problem 8.16 (as modification of S97,MR)
For any real $x_1, x_2, ..., x_n$ such that $x_1 + x_2 + ... + x_n = n$ prove inequality

$$x_1^2 x_2^2 ... x_n^2 \left(x_1^2 + x_2^2 + ... + x_n^2\right) \le n.$$

★Problem 8.17 (W6, J. Wildt IMO, 2014)
Let D_1 be set of strictly decreasing sequences of positive real numbers with first term equal to 1.For any $\mathbf{x}_{\mathbb{N}} := (x_1, x_2, ..., x_n, ...) \in D_1$ prove that

$$\sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} \ge \frac{4}{9}$$

and find the sequence for which equality occurs.

★ Problem 8.18 (SSMJ 5345)
Let $a, b > 0$. Prove that for any x, y holds inequality

$$|a \cos x + b \cos y| \le \sqrt{a^2 + b^2 + 2ab \cos(x + y)}$$

and find when equality occurs.

★Problem 8.19
For any natural n and m prove inequality

$$\left(n^m + n^{m-1} + ... + n + 1\right)^n \ge (m + 1)^n (n!)^m.$$

★Problem 8.20
Prove that $(n + 1) \cos \dfrac{\pi}{n + 1} - n \cos \dfrac{\pi}{n} > 1$ for any natural $n \ge 2$.

Finding maximum,minimum and range.

Problem 8.21 (82-Met. Rec.)

Find the min $\dfrac{-x^2+2x-1}{6x^2-7x+3}$ without using derivative.

Problem 8.22 (83-Met. Rec.)

Let $S(x,y) \ := \min\left\{x, \dfrac{1}{y}, y+\dfrac{1}{x}\right\}$ where x, y be positive real numbers.
Find $\max\limits_{x,y} S(x,y)$.

★Problem 8.23(58-Met. Rec.).

Find the maximal value of remainder from division of natural number n by natural number a, where $1 \leq a \leq n$ ($\max\limits_{1\leq a\leq n} r_a(n), n \in \mathbb{N}$).

★Problem 8.24

Find $\min\limits_{x,y,z} F(x,y,z)$,where $F(x,y,z) = \max\{|\cos x\ | + |\cos 2y|,\ \ |\cos y\ | + |\cos 2z|,\ \ |\cos z\ | + |\cos 2x|\}$.

Problem 8.25 (73-Met. Rec.)(M1067, ZK)

Let x, y, z be positive real numbers such that $xy+yz+zx=1$.
Find the minimal value of expression $\dfrac{x}{1-x^2}+\dfrac{y}{1-y^2}+\dfrac{z}{1-z^2}$.

★Problem 8.26 (SSMJ 5404)**

For any given positive integer $n \geq 3$ find smallest value of product $x_1x_2...x_n$ where $x_1, x_2, ..., x_n > 0$ and $\dfrac{1}{1+x_1}+\dfrac{1}{1+x_2}+...+\dfrac{1}{1+x_n}=1$.

9 Invariants.

Problem 9.1 (65-Met. Rec.).

a) An arbitrary fraction $\dfrac{a}{b}$ may be replaced by one of the fractions $\dfrac{a-b}{b}, \dfrac{a+b}{b}, \dfrac{b}{a}$.
Is it possible that after several such transformation starting with fraction $1/2$ obtain the fraction $67/91$?

b) An arbitrary pair of fraction $\left(\dfrac{a}{b}, \dfrac{c}{d}\right)$may be replaced by one the following pairs of fractions
$\left(\dfrac{a+b}{b}, \dfrac{c+d}{d}\right), \left(\dfrac{a-b}{b}, \dfrac{c-d}{d}\right), \left(\dfrac{b}{a}, \dfrac{d}{c}\right)$.
Is it possible that after several such transformation starting with the pair $(1/2, 3/4)$ obtain the the pair $(5/6, 9/11)$?

c) Given the triple of number $\left(2, \sqrt{2}, 1/\sqrt{2}\right)$ .Allowed any two numbers from current triple (a,b,c) replace with their sum divided by $\sqrt{2}$ and their difference
divided by $\sqrt{2}$. Is it possible after some numbers of allowed transformations obtain the triple $\left(1, \sqrt{2}, \sqrt{2}-1\right)$.

Problem 9.2 (66-Met. Rec.).

On Rainbow Island living 13 red, 15 green, 17 yellow chameleons. When two chameleons of one color meet each other then nothing happens, but if they have different color, they both change the color to the third one. Is it possible that with time all chameleons on island became of one color?

Problem 9.3
In the box are 13 red and 17 white balls. Permitted in any order and any number of the following operations:
1. Remove from the box one red ball and put it in a box two white balls;
2. Put it in a box one red ball and two white balls;
3. Remove from the box two red balls and put it in a box one white ball;
4. Remove from the box one red ball and two white balls.
Is it possible that after some number of permitted operations to lay in the box 37 red and 43 white balls?

9.1 Miscellaneous problems.

Problem 10.1 (1-Met. Rec.)
The 8 pupils bring from forest 60 mushrooms. Neither two from them bring mushrooms equally. Prove that among those pupils has three pupils, whose collect amount of mushrooms not less than the other five pupils.

Problem 10.2 (2-Met. Rec.)
2000 apples lies in several baskets. Permitted to remove the basket and removing any number of the apples from baskets.
Prove it's possible to obtain situation that in all baskets that remains numbers of apples are equal and common number of apples would be not less then 100.

Problem 10.3 (3-Met. Rec.)
Prove that digit of tens in 3^n is even number.

Problem 10.4 (7-Met. Rec.)
Does exist natural number such that first 8 digits after decimal dot of $\sqrt{n}$ are 19851986?

Problem 10.5 (12-Met. Rec.)
Prove that if $2a+3b+6c=0, a\neq 0$ then quadratic equation $ax^2+bx+c=0$ has at least one root on the interval $(0,1)$.

Problem 10.6 (13-Met. Rec.)

Prove that if $a\left(4a+2b+c\right)<0$ then $b^2>4ac$.

Problem 10.7 (18-Met. Rec.)
Prove that derivative of function
$$f\left(x\right)=\frac{x-1}{x-2}\cdot\frac{x-3}{x-4}\cdot\ldots\cdot\frac{x-2n+1}{x-2n}$$
is negative in all points of domain of $f\left(x\right)$.

Problem 10.8 (20-Met. Rec.)
Is it always from the sequence of n^2+1-th numbers $a_1,a_2,...,a_{n^2},a_{n^2+1}$ is possible to select a monotonous subsequence of lenght $n+1$?

Problem 10.9 (22-Met. Rec.)
Let natural numbers n,m satisfy inequality $\sqrt{7}-\dfrac{m}{n}>0$. Then holds inequality
$$\sqrt{7}-\frac{m}{n}>\frac{1}{mn}$$
As variant : Find max $\left\{m^2-7n^2 \mid m,n\in\mathbb{N} \text{ and } \dfrac{m}{n}<\sqrt{7}\right\}$.

Problem 10.10 (39-Met. Rec.)
Rational number represented by irreducible fraction $\dfrac{p}{q}$ belong to interval $\left(\dfrac{6}{13},\dfrac{7}{15}\right)$. Prove that $q\geq 28$.

★Problem 10.11
Find all one hundred digits numbers such that each of them equal to sum that addends are all its digits, all pairwise products of its digits and so on,... and at last product of all its digits.

Problem 10.12 (51-Met. Rec.)
Let $P\left(x\right)$ be polynomial with integer coefficients. Known that $P\left(0\right)$ and $P\left(1\right)$ are odd numbers. Prove that $P\left(x\right)$ have no integer roots.

Problem 10.13 (52-Met. Rec.)
Known that value of polynomial $P\left(x\right)$ with integer coefficients in three different points equal to 1. Is it possible that $P\left(x\right)$ has integer root?

Problem 10.14 (53-Met. Rec.)
Let $P\left(x\right)$ be polynomial with integer coefficients and $P\left(n\right)=m$ for some integer $n,m(m\neq 0)$. Then $P\left(n+km\right)$ divisible by m for any natural k.

Problem 10.15 (61-Met. Rec.)
Find the composition $g\left(x\right)=\underbrace{f\left(f\left(...f\left(x\right)...\right)\right)}_{n-\text{times}}$, where

a) $f(x) = \dfrac{x}{\sqrt{1-x^2}}$;

b) $f(x) = \dfrac{x\sqrt{3}-1}{x+\sqrt{3}}$.

Problem 10.16 (62-Met. Rec.)
Let $F(x) = \dfrac{4^x}{4^x+2}$.Find $F\left(\dfrac{1}{1988}\right) + F\left(\dfrac{2}{1988}\right) + ... + F(1)$.

Problem 10.17(63-Met. Rec.)
Let $f(q)$ the only root of the cubic equation $x^3 + px - q = 0$,where p is given positive real number. Prove that $f(q)$ is increasing function in $q \in \mathbb{R}$.

Problem 10.18 (64-Met. Rec.)
Let $P(x)$ be a polynomial such that equation $P(x) = x$ have no roots.
Is there a root of the equation $P(P(x)) = x$?

Problem 10.19 (67-Met. Rec).
The two rows of boys and girls set (in the first row, all boys, all girls in the second row), so that against every girl stand the boy that not lower than girl, or differs by the growth from her not more than 10 cm.
Prove that if children positioned in the each row accordingly their growth then against each girl will be a boy which again not lower than girl, or differs by the growth from her not more than 10 cm.

Problem 10.20 (86-Met. Rec.)
Find all values of real parameter b for which system
$$\begin{cases} x \geq (y-b)^2 \\ y \geq (x-b)^2 \end{cases}$$
has only solution.

★Problem 10.21 (CRUX 3090)
Find all non-negative real solutions (x, y, z) to the following system of inequalities:
$$\begin{cases} 2x(3-4y) \geq z^2+1 \\ 2y(3-4z) \geq x^2+1 \\ 2z(3-4x) \geq y^2+1 \end{cases}.$$

★Problem 10.22 (87-Met.Rec.)
Let A_1, A_2, A_3, A_4 be consequtive points on a circle and let a_i is number of rings on the rod at the point $A_i = 1, 2, 3, 4$. Find the maximal value of $2-$rings chains, that can be constructed from rings taken by one from any 2 neighboring, staying in cyclic order, rods.

Problem 10.23 (Problem with light bulbs).
n light bulbs together with its switches initially turned off arranged in a row and numbered from left to right consequtively by numbers from 1 to n.

If you click to the k-th switch than all light bulbs staying on the places numbered by multiples of k change state (turned off, turned on).
Some person moving from left to right along a row of light bulbs switch clicks each bulb (once). How many bulbs will light up when he comes to the last light bulb.

★Problem 10.24 (O274, MR4,2013).
Let a, b, c nonnegative integer numbers such that a and b are relatively prime. How many lattice points belong to domain
$D := \{(x, y) \mid x, y \in \mathbb{Z}, x, y \geq 0 \text{ and } bx + ay \leq abc\}$.

Problem 10.25 (102.-Met. Rec.)
Let α be irrational number. Prove that following function $f(x)$ is non periodic:
a) $f(x) = \sin \alpha x + \sin x$;
b) $f(x) = \sin \alpha x + \cos x$;
c) $f(x) = \tan \alpha x + \tan x$;
d) $f(x) = \tan \alpha x + \sin x$.

Problem 10.26 (103.-Met.Rec)

Let $a_1 = \dfrac{1}{2}, a_{n+1} = a_n + a_n^2, n \in \mathbb{N}$. For any $n \geq 2$ determine
$\left\lfloor \dfrac{1}{a_1+1} + \dfrac{1}{a_2+1} + \dots + \dfrac{1}{a_n+1} \right\rfloor$.

Problem 10.27 (Austria – Poland, 1980).
Given numerical sequence which for any $k, m \in \mathbb{N}$ satisfies to inequality
$|a_{m+k} - a_k - a_m| \leq 1$.
Prove that for any $p, q \in \mathbb{N}$ holds inequality $\left| \dfrac{a_p}{p} - \dfrac{a_q}{q} \right| < \dfrac{1}{p} + \dfrac{1}{q}$.

Problem 10.28 (M.1195 ZK Proposed by ,Proposed by O.T.Izhboldin)

Prove that if sequence (a_n) satisfied to condition $|a_{n+m} - a_n - a_m| \leq \dfrac{1}{n+m}$, then (a_n) is arithmetic progression.

★Problem 10.29 (3571,CRUX,2010)
For given natural $n \geq 2$, among increasing arithmetic progression $x_1, x_2, \dots, x_n$ such that $x_1^2 + x_2^2 + \dots + x_n^2 = 1$, find arithmetic progression with greates common difference d.

Problem 10.30.Quickies-Q2(CRUX?)
What is the units digit of the real number
$\left(15 + \sqrt{220}\right)^{2004} + \left(15 + \sqrt{220}\right)^{2005}$?

